

Quantum field theoretic properties of Lorentz-violating operators of nonrenormalizable dimension in the fermion sector

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In a recent article the nonminimal fermion sector of the Lorentz-violating Standard-Model Extension was introduced and its operators were classified according to their properties with respect to charge conjugation C and the discrete Lorentz transformations P , T . The current paper deals with the quantum field theory based on certain sets of these nonminimal coefficients. In particular, three families of coefficients are considered where two of them are CPT -even and the third is CPT -odd. The modified fermion dispersion relations are obtained plus the positive- and negative-energy solutions of the modified Dirac equation. Using these solutions the spinor completeness relations are calculated and, besides that, the fermion propagator is derived. These are used to demonstrate the validity of the optical theorem at tree-level, which provides a cross check for the results obtained. For the first two cases considered the proof is exact in the Lorentz-violating coefficients whereas for the third case it is restricted to first order. The results provide insight into the quantum-field theoretic properties of the nonminimal fermion sector, which will also be important for future phenomenological investigations.

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I. INTRODUCTION

Investigating the violation of Lorentz invariance has become more and more attractive in the recent years. A possible violation of this fundamental symmetry of the laws of nature is motivated by physics at the Planck scale such as string theory [1–4], loop quantum gravity [5, 6], field theory on noncommutative spacetimes [7], spacetime foam models [8, 9] plus quantum field theory on spacetimes with a nontrivial topology [10, 11]. An effective description of Lorentz symmetry violation for energies that are much smaller than the Planck scale is provided by the Standard-Model Extension (SME) [12]. The latter forms a test framework for experimental searches [13] for Lorentz symmetry violation and it allows one to investigate the properties of quantum field theories based on certain sectors of this framework. The SME includes all operators of Standard Model fields that are invariant with respect to the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ and violate particle Lorentz invariance. The minimal version is restricted to only power-counting renormalizable operators, whereas the nonminimal version also contains operators of higher dimension [14–16].

In a series of articles quantum field theories based on a Lorentz-violating modification of the minimal photon sector were examined [17–23]. They applied to characteristics of the modified photon propagator, the polarization vectors, unitarity, and microcausality. In a recent paper these methods are even employed to the quantum field theory based on a special set of operators of the nonminimal photon sector [24]. Although there is more knowledge to be gained for the nonminimal photon sector, some of the main properties of a quantum field theory based on an isotropic operator of nonrenormalizable dimension were obtained in the latter reference. To extend the picture the current paper is devoted to similar investigations for the nonminimal fermion sector. Note that the majority of both experimental and theoretical investigations performed to date is restricted to the minimal fermion sector of the SME.

The basis of a quantum field theory of spin-1/2 particles is formed by the Dirac equation. Dirac introduced the equation that is named after him in 1928 for several reasons [25]. First, the number of stationary states in hydrogen-like atoms were observed to be twice as what the quantum theory of a pointlike electron without internal quantum numbers would suggest. To account for this doubling of states the quantum-mechanical spin was introduced by Pauli and Darwin (see [25] and references therein). However it was unsatisfactory that the spin had to be introduced by hand and did not arise naturally from the theory. Second, the relativistic wave equation proposed by Klein and Gordon evidently would describe electrons of both negative and positive charge where the latter are associated with a negative energy. Classically these solutions could be discarded but quantum mechanically transitions between states with negative and positive charge could be induced by perturbation, which is not observed in nature. These, amongst other problems, were solved by the Dirac equation, which incorporates special relativity into quantum mechanics and, therefore, naturally describes the electron spin. Beyond the context of quantum mechanics it was reinterpreted and used in quantum field theory to describe fermions with spin 1/2.

A modified version of the Dirac Lagrange density leading to a modified Dirac equation forms the foundation of the Lorentz-violating fermion sector. Quantum-field theoretic properties of the minimal fermion sector such as microcausality and stability were investigated in [26]. Furthermore in [27] the implications of a nonvanishing torsion coupling to the fermion sector were considered where the occurring terms were stated, classified, and embedded into the minimal fermionic

SME. Therefore already existing bounds on minimal fermionic coefficients could be reinterpreted as bounds on the torsion tensor coefficients.

Note that certain coefficients of the minimal SME fermion sector are not observable in Minkowski spacetime, even if they lie several orders of magnitudes above the current experimental bounds of observable coefficients. In [28] it was shown that in the presence of gravity the fluctuational part of a special sample of coefficients can be detected if they couple to the gravitational field. This was then exploited to obtain several bounds on these coefficients from experiment.

The gravitational interaction on the Lorentz-violating minimal fermion sector was extensively studied in [29] where a large number of bounds was determined on the coefficients. This list of bounds was extended in [30] by considering an additional experimental setup that had a priori not been designed for experiments in a gravitational background. Furthermore, in [31, 32] it is proposed how fermionic Lorentz-violating coefficients can be constrained by antimatter tests in gravitational physics.

The current paper is organized as follows. Section II provides the action of the nonminimal free SME fermion sector and restricts it to the operators that shall be investigated throughout the paper. In Sec. III the modified fermion dispersion laws will be examined and Sec. IV is dedicated to the properties of the modified Dirac spinors. Section V deals with the fermion propagator and the optical theorem at tree-level, which relates the propagator to the spinor completeness relations. Finally, in Sec. VI the analysis is extended to alternative sets of Lorentz-violating operators of the nonminimal fermion sector. The results are summarized and discussed in Sec. VII. Computational details are presented in the appendix. Throughout the article natural units with $\hbar = c = 1$ will be used unless stated otherwise.

II. INTRODUCTION OF THE THEORY

The theory considered is a Lorentz-violating extension of the free Standard Model fermion sector [15], which is based on the following action:

$$S = \int d^4x \mathcal{L}, \quad \mathcal{L} = \frac{1}{2} \bar{\psi} \left(\gamma^\mu i \partial_\mu - m_\psi \mathbb{1}_4 + \hat{\mathcal{Q}} \right) \psi + \text{h.c.}, \quad (2.1)$$

with the standard Dirac field ψ , the Dirac conjugate field $\bar{\psi} = \psi^\dagger \gamma^0$, the fermion mass m_ψ , and the unit matrix $\mathbb{1}_4$ in spinor space. The standard gamma matrices γ^μ for $\mu = 1 \dots 4$ satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}_4$ with the Minkowski metric $(\eta^{\mu\nu}) = \text{diag}(1, -1, -1, -1)$. The part $\hat{\mathcal{Q}}$ contains all Lorentz-violating operators to arbitrary operator dimension that are compatible with the fermion sector. The Lagrange density is written such that the corresponding Hamilton operator is Hermitian.

A. Scalar family of coefficients

The Lorentz-violating coefficients linked to the modification $\hat{\mathcal{Q}}$ can be grouped in different classes according to their properties under (proper) observer Lorentz transformations plus C , P , and T transformations. This was done in Tab. I of [15]. The simplest family of coefficients is undoubtedly the scalar \hat{m} in the first line of the latter table. Therefore, first of all the action of Eq. (2.1) will

be restricted to this particular subset of Lorentz-violating coefficients. The parameter family \hat{m} is *CPT*-even and does not have a dimension-4 operator equivalent. Then \hat{Q} is given by

$$\hat{Q} = -\hat{m}\mathbb{1}_4, \quad \hat{m} = \sum_{\substack{d \text{ odd} \\ d \geq 5}}^{\infty} m^{(d)\alpha_1 \dots \alpha_{(d-3)}} p_{\alpha_1} \dots p_{\alpha_{(d-3)}}. \quad (2.2)$$

In [15] these expansions are directly defined in momentum space. Then no additional signs have to be taken into account, which simplifies the notation. The family \hat{m} has mass dimension 1 and via the given expansion it is decomposed into coefficients that have a mass dimension smaller than 1 and are contracted with additional four-momenta. Note that the number d does not give the mass dimension of the coefficients but the dimensionality of the corresponding Lorentz-violating operator. Restricting to the set of 10 coefficients $m^{(5)\alpha_1\alpha_2}$ having mass dimension -1 we have to consider

$$\hat{m} = \hat{m}(p^0, \mathbf{p}) = m^{(5)\alpha_1\alpha_2} p_{\alpha_1} p_{\alpha_2}. \quad (2.3)$$

The arguments of \hat{m} will be suppressed unless caution is required. The advantage of using this subset of coefficients is that we can solely concentrate on the effects that are characteristic for higher-dimensional operators and that do not have an equivalent for marginal operators (operators with dimension ≤ 4). It is reasonable to follow the same steps as in [24] and to consider three different sectors of the ten coefficients:

$$m^{(5)} = \left(\begin{array}{c|cccc} m^{(5)00} & m^{(5)01} & m^{(5)02} & m^{(5)03} \\ \hline m^{(5)01} & m^{(5)11} & m^{(5)12} & m^{(5)13} \\ m^{(5)02} & m^{(5)12} & m^{(5)22} & m^{(5)23} \\ m^{(5)03} & m^{(5)13} & m^{(5)23} & m^{(5)33} \end{array} \right). \quad (2.4)$$

The sector consisting of the single coefficient $m^{(5)00}$ will be called “temporal,” the sector made up of the three coefficients $m^{(5)0i}$ for $i = 1 \dots 3$ will be named “mixed” and the set of the remaining coefficients $m^{(5)ij}$ for $i, j = 1 \dots 3$ will be denoted as “spatial.”

III. MODIFIED FERMION DISPERSION LAWS

In the current section the modified fermion dispersion relations shall be computed and their properties will be discussed. Equation (39) with the definition (35) in [15] states the general off-shell dispersion relation for the Lorentz-violating fermion sector defined by the action of Eq. (2.1). For the special case considered here we have that $\hat{\mathcal{S}}_{\pm} = -(m_{\psi} + \hat{m})$, $\hat{\mathcal{V}}_{\pm}^{\mu} = p^{\mu}$, $\hat{\mathcal{T}}_{\pm}^{\mu\nu} = 0$ and the off-shell dispersion law reads

$$p^2 - (m_{\psi} + \hat{m})^2 = 0, \quad (3.1)$$

where $p = (p^0, \mathbf{p}) \equiv (\tilde{E}_{\psi}, \mathbf{p})$ is the fermion four-momentum with the spatial momentum \mathbf{p} . The zeros of Eq. (3.1) with respect to \tilde{E}_{ψ} correspond to the modified dispersion relations of a fermion. There are both zeros $\tilde{E}_{\psi}^{(>)} > 0$ and $\tilde{E}_{\psi}^{(<)} < 0$ where only the positive-energy solutions will be given in what follows. For the temporal sector they read:

$$\tilde{E}_{\psi;1,2}^{(\text{temp})} = \frac{\sqrt{1 - 2m^{(5)00}m_{\psi}} \mp \sqrt{1 - 4m^{(5)00}(m_{\psi} + m^{(5)00}\mathbf{p}^2)}}{\sqrt{2}|m^{(5)00}|}, \quad (3.2a)$$

with the expansions

$$\tilde{E}_{\psi;1}^{(\text{temp})} = \sqrt{\mathbf{p}^2 + m_\psi^2} (1 + m^{(5)00} m_\psi) + \mathcal{O}[(m^{(5)00})^2], \quad (3.2b)$$

$$\tilde{E}_{\psi;2}^{(\text{temp})} = \frac{1}{m^{(5)00}} - m_\psi - \frac{1}{2}(\mathbf{p}^2 + 2m_\psi^2)m^{(5)00} + \mathcal{O}[(m^{(5)00})^2]. \quad (3.2c)$$

Hence there are two modified dispersion laws. The first is a perturbation of the standard dispersion relation for a Dirac fermion with spatial momentum \mathbf{p} and mass m_ψ . However the second does evidently not have a limit for vanishing Lorentz-violating coefficient $m^{(5)00}$. Instead there is an energy gap, which is inversely proportional to the coefficient $m^{(5)00}$. The latter dispersion law may become important for large fermion momentum indicating that it is related to Planck scale physics. Such dispersion relations can be considered as spurious for momenta that are much smaller than the Planck scale. They also appear in the context of the nonminimal photon sector (cf. [14, 24]) and how to deal with them will be described later.

For the mixed sector one obtains:

$$\tilde{E}_\psi^{(\text{mixed})} = \frac{\mathbf{p}^2 + m_\psi^2}{\sqrt{[1 - (\hat{m}_1)^2] \mathbf{p}^2 + m_\psi^2 + \hat{m}_1 m_\psi}}, \quad \hat{m}_1 = 2m^{(5)0i} p^i, \quad (3.3a)$$

$$\tilde{E}_\psi^{(\text{mixed})} = \sqrt{\mathbf{p}^2 + m_\psi^2} - \hat{m}_1 m_\psi + \mathcal{O}[(\hat{m}_1)^2]. \quad (3.3b)$$

Here no spurious dispersion law appears in contrast to the mixed sector of the particular set of nonminimal photon coefficients considered in [24]. Last but not least, for the spatial sector the modified dispersion law is given by:

$$\tilde{E}_\psi^{(\text{spatial})} = \sqrt{\mathbf{p}^2 + (m_\psi + \hat{m}_2)^2}, \quad \hat{m}_2 = m^{(5)ij} p^i p^j, \quad (3.4a)$$

$$\tilde{E}_\psi^{(\text{spatial})} = \sqrt{\mathbf{p}^2 + m_\psi^2} \left(1 + \frac{m_\psi}{\mathbf{p}^2 + m_\psi^2} \hat{m}_2 \right) + \mathcal{O}[(\hat{m}_2)^2]. \quad (3.4b)$$

Also for the spatial sector there is no spurious dispersion relation in accordance to the nonminimal photon theory discussed in the latter reference.

For fermions the negative-energy solutions have a physical meaning as well. They will not be given explicitly but they are related to the positive-energy solutions as follows: $\tilde{E}_\psi^{(>)}(\mathbf{p}, m^{(5)\alpha_1\alpha_2}) = -\tilde{E}_\psi^{(<)}(-\mathbf{p}, m^{(5)\alpha_1\alpha_2})$. According to the Feynman-Stückelberg interpretation a negative-energy particle propagating backwards in time, i.e., having four-momentum $(p^\mu) = (-p^0, -\mathbf{p})^T$, can be interpreted as a positive-energy antiparticle propagating forwards in time with $(p^\mu) = (p^0, \mathbf{p})^T$. Hence after reinterpreting $p^0 = \tilde{E}_\psi^{(<)}(\mathbf{p}, m^{(5)\alpha_1\alpha_2})$ with $p^\mu \mapsto -p^\mu$, the latter transforms to $p^0 = \tilde{E}_\psi^{(>)}(\mathbf{p}, m^{(5)\alpha_1\alpha_2})$. While $\tilde{E}_\psi^{(>)}(\mathbf{p}, m^{(5)\alpha_1\alpha_2})$ is the energy of a spin-1/2 matter particle, this can also be understood as the energy of the corresponding antimatter particle. Since the coefficients \hat{m} , which are closely linked to the fermion mass m_ψ , are *CPT*-even [15], their sign is not reversed when considering the negative-energy solutions. As a result, the particle and antiparticle energies are equal. This is in accordance to the corresponding rules for the minimal fermion sector [26]. Note that if we took the spurious dispersion law of Eq. (3.2c) into account for both particles and antiparticles, the Lorentz-violating coefficient $m^{(5)00}$ would be restricted to positive values.

It can be checked that the expansions of Eqs. (3.2b), (3.3b), and (3.4b) at first order in Lorentz violation are in agreement with the upper 2×2 block of Eq. (59) in [15] for particles and the reinterpreted lower 2×2 block for antiparticles.

IV. MODIFIED DIRAC SPINORS

The Lagrange density in Eq. (2.1) leads to a modified Dirac equation for the spinor field ψ that is given as follows:

$$(\not{p} - m_\psi \mathbb{1}_4 + \hat{\mathcal{Q}})\psi = 0, \quad (\gamma^\mu) = (\gamma^0, \boldsymbol{\gamma})^T, \quad \boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^3)^T. \quad (4.1)$$

After having obtained the modified fermion dispersion laws in the last section the solutions of the modified Dirac equation will be determined. According to [15] we choose a special representation of gamma-matrices — the chiral representation, in which the $\gamma^{0,1,2,3}$ are block off diagonal and γ^5 is diagonal. Explicitly the matrices are given by:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^{1,2,3} = \begin{pmatrix} 0 & \sigma^{1,2,3} \\ -\sigma^{1,2,3} & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \quad (4.2a)$$

with the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (4.2b)$$

and the two-dimensional unit matrix $\mathbb{1}_2$. For the particular case considered the solutions of the Dirac equation can be determined from Eqs. (51), (53), and (54) in [15]. The procedure needed will be reviewed as follows. First of all, a unitary matrix U depending on an energy scale $E \geq 0$ and a mass scale m has to be constructed:

$$U(E, m, \mathbf{p}) = V \cdot W(E, m, \mathbf{p}), \quad V = \frac{\mathbb{1}_4 + \gamma^0\gamma^5}{\sqrt{2}}, \quad W(E, m, \mathbf{p}) = \frac{(E + m)\mathbb{1}_4 + \mathbf{p} \cdot \boldsymbol{\gamma}}{\sqrt{2E(E + m)}}. \quad (4.3)$$

For the standard theory with zero Lorentz violation, m corresponds to the fermion mass m_ψ and E to the fermion energy $E_\psi = \sqrt{\mathbf{p}^2 + m_\psi^2}$. Using the matrix U the Dirac equation can be diagonalized leading to the following eigenvalue problem for the fermion energy E :

$$(E\mathbb{1}_4 - H)U\psi = 0, \quad H = -\gamma_5 E = \begin{pmatrix} E_\psi \mathbb{1}_2 & 0 \\ 0 & -E_\psi \mathbb{1}_2 \end{pmatrix}. \quad (4.4)$$

According to [15], Eq. (4.4) can still be used to write up the spinors that are a solution of the modified Dirac equation of Eq. (4.1). The positive-energy spinors u can then be obtained as follows:

$$u^{(\alpha)}(\tilde{E}_\psi^{(>)}, \mathbf{p}) = \frac{1}{\sqrt{N_u^{(\alpha)}}} U^\dagger(\tilde{E}_\psi^{(>)}, \tilde{m}_\psi, \mathbf{p}) u^{(\alpha)}(\tilde{m}_\psi, \mathbf{0}), \quad (4.5a)$$

$$u^{(1)}(\tilde{m}_\psi, \mathbf{0}) = \begin{pmatrix} \phi^{(1)} \\ \mathbf{0} \end{pmatrix}, \quad u^{(2)}(\tilde{m}_\psi, \mathbf{0}) = \begin{pmatrix} \phi^{(2)} \\ \mathbf{0} \end{pmatrix}, \quad \phi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (4.5b)$$

$$\tilde{m}_\psi \equiv m_\psi + \hat{m}(\tilde{E}_\psi^{(>)}, \mathbf{p}). \quad (4.5c)$$

where $\alpha = 1, 2$ and $\tilde{E}_\psi^{(>)}$ is the positive fermion energy that is modified due to Lorentz violation (cf. Eqs. (3.2a), (3.3a), and (3.4a), respectively, for the three different sectors considered). The spinors are a solution of the modified Dirac equation: $(\not{p} - \tilde{m}_\psi \mathbb{1}_4)u^{(\alpha)}(p) = 0$ with $p^0 = \tilde{E}_\psi^{(>)}$. The normalization $N_u^{(\alpha)}$ of the spinors is chosen such that

$$\bar{u}^{(\alpha)}(p)u^{(\beta)}(p) = u^{(\alpha)\dagger}(p)\gamma^0 u^{(\beta)}(p) = 2\tilde{m}_\psi \delta^{\alpha\beta}, \quad (4.6a)$$

$$u^{(\alpha)\dagger}(p)u^{(\beta)}(p) = 2\tilde{E}_\psi^{(>)} \delta^{\alpha\beta}. \quad (4.6b)$$

Furthermore the associated modified spinor completeness relation reads

$$\sum_{\alpha=1,2} u^{(\alpha)}(p)\bar{u}^{(\alpha)}(p) = \not{p} + \tilde{m}_\psi \mathbb{1}_4. \quad (4.7)$$

On the other hand, the negative-energy spinors are given by

$$v^{(\alpha)}(\tilde{E}_\psi^{(>)}, \mathbf{p}) = \frac{1}{\sqrt{N_v^{(\alpha)}}} U^\dagger(\tilde{E}_\psi^{(>)}, \tilde{m}_\psi, -\mathbf{p}) v^{(\alpha)}(\tilde{m}_\psi, \mathbf{0}), \quad (4.8a)$$

$$v^{(1)}(\tilde{m}_\psi, \mathbf{0}) = \begin{pmatrix} \mathbf{0} \\ \chi^{(1)} \end{pmatrix}, \quad v^{(2)}(\tilde{m}_\psi, \mathbf{0}) = \begin{pmatrix} \mathbf{0} \\ \chi^{(2)} \end{pmatrix}, \quad \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (4.8b)$$

$$\tilde{m}_\psi = m_\psi + \hat{m}(-\tilde{E}_\psi^{(>)}, -\mathbf{p}) = m_\psi + \hat{m}(\tilde{E}_\psi^{(>)}, \mathbf{p}). \quad (4.8c)$$

Note the minus signs associated to the four-momentum components p^0 and \mathbf{p} . These spinors are a solution of the modified Dirac equation $(\not{p} - \tilde{m}_\psi \mathbb{1}_4)v^{(\alpha)}(p) = 0$ with $(p^\mu) = (-\tilde{E}_\psi^{(>)}, -\mathbf{p})^T$. Here the normalization $N_v^{(\alpha)}$ is chosen so that the following relationships hold:

$$\bar{v}^{(\alpha)}(p)v^{(\beta)}(p) = v^{(\alpha)\dagger}(p)\gamma^0 v^{(\beta)}(p) = -2\tilde{m}_\psi \delta^{\alpha\beta} \quad (4.9a)$$

$$v^{(\alpha)\dagger}(p)v^{(\beta)}(p) = -2\tilde{E}_\psi^{(>)} \delta^{\alpha\beta}. \quad (4.9b)$$

For the negative-energy spinors the modified completeness relation is given by

$$\sum_{\alpha=1,2} v^{(\alpha)}(p)\bar{v}^{(\alpha)}(p) = \not{p} - \tilde{m}_\psi \mathbb{1}_4. \quad (4.10)$$

On the right-hand sides of the completeness relations of Eqs. (4.7) and (4.10), $(p^\mu) = (\tilde{E}_\psi^{(>)}, \mathbf{p})^T$ is understood. An explicit derivation of all these relations can be found in App. A 1.

V. MODIFIED FERMION PROPAGATOR AND THE OPTICAL THEOREM

Having obtained the modified spinors and completeness relations in the last section, the fermion propagator will be computed in what follows. The fermion propagator $S(p)$ is the inverse (modulo a factor of i) of the operator $S^{-1}(p) \equiv \not{p} - m_\psi \mathbb{1}_4 + \hat{\mathcal{Q}}$ that appears in the modified Dirac equation:

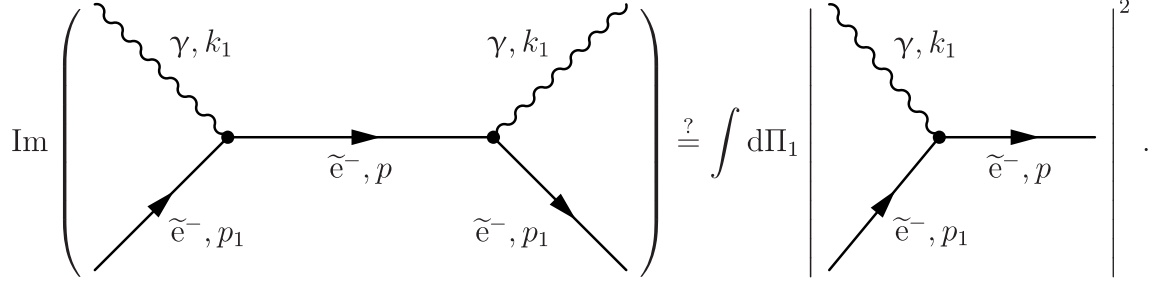


FIG. 1: Forward scattering amplitude of tree-level electron photon scattering that is related to the total cross section of electron photon scattering, if the optical theorem is valid. A modified electron is denoted by \tilde{e}^- and a photon by γ . The momenta are noted next to the particle symbols and the one-particle phase space is called $d\Pi_1$.

$S(p)S^{-1}(p) = S^{-1}(p)S(p) = i\mathbb{1}_4$. In this case $S^{-1}(p) = \not{p} - (m_\psi + \hat{m})\mathbb{1}_4$. From the latter equation the propagator can be determined and it is expressed in terms of gamma matrices as follows:

$$S(p) = \frac{i}{p^2 - (m_\psi + \hat{m})^2} [\not{p} + (m_\psi + \hat{m})\mathbb{1}_4]. \quad (5.1)$$

As a good cross check for the electron propagator of Eq. (5.1) and the spinor completeness relation of Eq. (4.7) the optical theorem can be used. Therefore we consider a modified electron \tilde{e}^- scattering at a standard photon γ (Compton scattering). The forward scattering amplitude at tree-level corresponds to the left-hand side of the equation shown in Fig. 1 and it is denoted by $\mathcal{M} \equiv \mathcal{M}(\tilde{e}^- \gamma \rightarrow \tilde{e}^- \gamma)$. If the optical theorem is valid, the forward scattering amplitude will be related to the total cross section of the process $\tilde{e}^- \gamma \rightarrow \tilde{e}^-$ at tree-level where a summation over the spins of the final electron has to be performed.¹ If the spin state of the initial electron is denoted as α and the polarization state of the initial photon as λ , the forward scattering amplitude reads:

$$\mathcal{M} = - \int \frac{d^4 p}{(2\pi)^4} \delta^{(4)}(k_1 + p_1 - p) e^2 \bar{u}^{(\alpha)}(p_1) \gamma^\nu \frac{\not{p} + \tilde{m}_\psi \mathbb{1}_4}{p^2 - \tilde{m}_\psi^2 + i\epsilon} \gamma^\mu u^{(\alpha)}(p_1) \varepsilon_\mu^{(\lambda)}(k_1) \bar{\varepsilon}_\nu^{(\lambda)}(k_1). \quad (5.2)$$

Here $u^{(\alpha)}(p_1)$ is a positive-energy spinor describing a modified electron in the spin state α and with four-momentum p_1 . These were obtained in the previous section and are given by Eq. (4.5). The elementary charge is e and the Feynman propagator poles are treated with the usual $i\epsilon$ -prescription. The polarization vector of a standard photon with polarization state λ and momentum k_1 is named $\varepsilon_\mu^{(\lambda)}(k_1)$. Total energy-momentum conservation of the process is encoded in the four-dimensional δ -function.

The interest lies in the imaginary part of Eq. (5.2). First of all only the mixed and the spatial sector of the theory, whose dispersion relations were obtained in Sec. III, are considered. These sectors are characterized by a single positive and a negative fermion energy and the denominator of the corresponding propagator can be factorized with respect to the poles as follows:

$$\frac{1}{p^2 - \tilde{m}_\psi^2 + i\epsilon} = \frac{1}{\left(p^0 - \tilde{E}_\psi^{(>)} + i\epsilon\right) \left(p^0 - \tilde{E}_\psi^{(<)} - i\epsilon\right)}, \quad (5.3)$$

¹ At tree-level Compton scattering has an additional contribution with the two vertices interchanged. The sum of both amplitudes is gauge invariant where a single contribution is not. Nevertheless to check the optical theorem we restrict ourselves to only the first contribution. If the optical theorem is valid, the imaginary part of the first amplitude will be related to the cross section of a physical process, which must be a gauge-invariant quantity. Hence the imaginary part of the corresponding forward scattering amplitude is then gauge-invariant, as well.

with the positive fermion energy $\tilde{E}_\psi^{(>)}$ and the negative-energy counterpart $\tilde{E}_\psi^{(<)}$. Due to energy-momentum conservation only the pole with a positive real part, i.e., $p^0 = \tilde{E}_\psi^{(>)} - i\epsilon$ contributes to the imaginary part. Interpreting the propagator as a distribution, one can use the following relation to treat the contributing pole where this procedure corresponds to applying Cutkosky's cutting rules [33]:

$$\frac{1}{p^0 - \tilde{E}_\psi^{(>)} + i\epsilon} = \mathcal{P} \frac{1}{p^0 - \tilde{E}_\psi^{(>)}} - i\pi \delta(p^0 - \tilde{E}_\psi^{(>)}) . \quad (5.4)$$

Here the first term involves the principal value \mathcal{P} , which is purely real. The second summand is imaginary and forces p^0 to be equal to the fermion energy $\tilde{E}_\psi^{(>)}$ in the integrand. With this input the imaginary part of Eq. (5.2) can be computed where additionally Eq. (4.7) is used:

$$\begin{aligned} 2\text{Im}(\mathcal{M}) &= \int \frac{d^3p}{(2\pi)^3 2\tilde{E}_\psi^{(>)}} \delta^{(4)}(k_1 + p_1 - p) e^2 \bar{u}^{(\alpha)}(p_1) \gamma^\nu (\not{p} + \tilde{m}_\psi \mathbb{1}_4) \gamma^\mu u^{(\alpha)}(p_1) \varepsilon_\mu^{(\lambda)}(k_1) \bar{\varepsilon}_\nu^{(\lambda)}(k_1) \\ &= \int \frac{d^3p}{(2\pi)^3 2\tilde{E}_\psi^{(>)}} \delta^{(4)}(k_1 + p_1 - p) \\ &\quad \times e^2 \bar{u}^{(\alpha)}(p_1) \gamma^\nu \left[\sum_{\beta=1,2} u^{(\beta)}(p) \bar{u}^{(\beta)}(p) \right] \gamma^\mu u^{(\alpha)}(p_1) \varepsilon_\mu^{(\lambda)}(k_1) \bar{\varepsilon}_\nu^{(\lambda)}(k_1) \\ &= \int \frac{d^3p}{(2\pi)^3 2\tilde{E}_\psi^{(>)}} \delta^{(4)}(k_1 + p_2 - p) \\ &\quad \times \sum_{\beta=1,2} \left(i e \bar{u}^{(\beta)}(p) \gamma^\nu u^{(\alpha)}(p_1) \varepsilon_\nu^{(\lambda)}(k_1) \right)^\dagger i e \bar{u}^{(\beta)}(p) \gamma^\mu u^{(\alpha)}(p_1) \varepsilon_\mu^{(\lambda)}(k_1) \\ &= \int \frac{d^3p}{(2\pi)^3 2\tilde{E}_\psi^{(>)}} \delta^{(4)}(k_1 + p_2 - p) \sum_{\beta=1,2} |\widehat{\mathcal{M}}|^2. \end{aligned} \quad (5.5)$$

Hence the imaginary part can be expressed with the matrix element $\widehat{\mathcal{M}} \equiv \mathcal{M}(\tilde{e}^- \gamma \rightarrow \tilde{e}^-)$ of the process on the right-hand side of the equation in Fig. 1. This shows that the optical theorem is valid for this particular process. Note that this proof is rather general and no relations were used that are supposedly only valid for the process considered.

An analogous computation can be done for spin-1/2 antimatter by considering the process $\tilde{e}^+ \gamma \rightarrow \tilde{e}^+ \gamma$, with a modified positron \tilde{e}^+ . Then the electron lines in the diagrams shown in Fig. 1 have to be replaced by positron lines. Since the momentum of the internal line flows in the opposite direction of the arrow on this line, the propagator momentum is now $-p^\mu$ instead of p^μ [34]. Furthermore a global factor of -1 has to be considered due to the interchange of fermionic operators when applying Wick's theorem:

$$\overline{\mathcal{M}} = \int \frac{d^4p}{(2\pi)^4} \delta^{(4)}(k_1 + p_1 - p) e^2 \bar{v}^{(\alpha)}(p_1) \gamma^\mu \frac{-\not{p} + \tilde{m}_\psi \mathbb{1}_4}{p^2 - \tilde{m}_\psi^2 + i\epsilon} \gamma^\nu v^{(\alpha)}(p_1) \varepsilon_\mu^{(\lambda)}(k_1) \bar{\varepsilon}_\nu^{(\lambda)}(k_1), \quad (5.6)$$

where $\overline{\mathcal{M}} \equiv \mathcal{M}(\tilde{e}^+ \gamma \rightarrow \tilde{e}^+ \gamma)$. Then the roles of the poles in Eq. (5.3) are interchanged where again the positive-energy pole is taken into account. A similar computation to what was done before together with the completeness relation (4.10) for the positron spinors leads to:

$$2\text{Im}(\overline{\mathcal{M}}) = \int \frac{d^3p}{(2\pi)^3 2\tilde{E}_\psi^{(>)}} \delta^{(4)}(k_1 + p_1 - p)$$

$$\begin{aligned}
& \times e^2 \bar{v}^{(\alpha)}(p_1) \gamma^\mu \left[\sum_{\beta=1,2} v^{(\beta)}(p) \bar{v}^{(\beta)}(p) \right] \gamma^\nu v^{(\alpha)}(p_1) \varepsilon_\mu^{(\lambda)}(k_1) \bar{\varepsilon}_\nu^{(\lambda)}(k_1) \\
& = \int \frac{d^3 p}{(2\pi)^3 2\widetilde{E}_\psi^{(>)}} \delta^{(4)}(k_1 + p_1 - p) \\
& \quad \times \sum_{\beta=1,2} i e \bar{v}^{(\alpha)}(p_1) \gamma^\mu v^{(\beta)}(p) \varepsilon_\mu^{(\lambda)}(k_1) \left(i e \bar{v}^{(\alpha)}(p_1) \gamma^\nu v^{(\beta)}(p) \varepsilon_\nu^{(\lambda)}(k_1) \right)^\dagger \\
& = \int \frac{d^3 p}{(2\pi)^3 2\widetilde{E}_\psi^{(>)}} \delta^{(4)}(k_1 + p_2 - p) \sum_{\beta=1,2} |\widetilde{\mathcal{M}}|^2, \tag{5.7}
\end{aligned}$$

with $\widetilde{\mathcal{M}} \equiv \mathcal{M}(\widetilde{e}^+ \gamma \rightarrow \widetilde{e}^+)$. Hence the validity of the optical theorem at tree-level can also be demonstrated for spin-1/2 antifermions. This is a good independent crosscheck for the modified spinors, the completeness relations, and the propagator. Since no relations were used that only hold for the particular process considered, this proof is rather general and valid for any tree-level process with an internal electron or positron line.

A last caveat is formed by the temporal sector of Sec. III. The latter is characterized by the two distinct positive-energy dispersion laws of Eq. (3.2a) where the first of them (and its negative-energy counterpart) is a perturbation of the standard one and the other is spurious. However the spurious solutions cannot simply be discarded when considering the optical theorem. In this case the propagator denominator of Eq. (5.3) has four distinct poles and the above proof has to be modified. Note that this issue also occurs in the context of the nonminimal *CPT*-even and isotropic modified Maxwell theory [24]. The problem may² occur if there are additional time derivatives in the Dirac operator leading to an unconventional time evolution for the Dirac field (see [35] for a related problem in the minimal fermion sector). In the minimal sector this is resolved by a field redefinition at first order Lorentz violation. This might be possible for the nonminimal case as well, but the approach introduced in [24] will be employed instead. By doing so, all additional p^0 components in the off-shell dispersion relation of Eq. (3.1) are replaced by the standard fermion dispersion law $p^0 = \sqrt{\mathbf{p}^2 + m_\psi^2}$: $m^{(5)00} p_0^2 = m^{(5)00} (\mathbf{p}^2 + m_\psi^2)$. The resulting expression is then still valid at first order Lorentz violation. Computing the modified positive-energy dispersion relation after the replacement has been performed, results in the only solution

$$\widetilde{E}_{\psi'}^{(\text{temp})} = \sqrt{(\mathbf{p}^2 + m_\psi^2) \left[1 + 2m^{(5)00} m_\psi^2 - (m^{(5)00})^2 (\mathbf{p}^2 + m_\psi^2) \right]}, \tag{5.8}$$

which coincides with Eq. (3.2a) at first order Lorentz violation. A spurious solution does not appear any more. Then all the steps of the above proof can be redone analogously and the optical theorem at tree-level is demonstrated to be valid at first order Lorentz violation for the temporal sector as well.

VI. APPLICATION TO OTHER SETS OF LORENTZ-VIOLATING COEFFICIENTS

In the previous sections certain properties of the quantum field theory based on the nonminimal Lorentz-violating set of fermion coefficients \widehat{m} were investigated and discussed. This particular

² For the nonminimal photon sector considered in [24] the issue appeared, if there was at least one additional time derivative. However in the context of the nonminimal fermion sector considered here there are no spurious dispersion relations for the mixed case of $m^{(5)\alpha_1\alpha_2}$ with only one additional derivative, for example.

family of coefficients was chosen in the first place because it is *CPT*-even and it forms a scalar under observer Lorentz transformations. Hence these parameters were supposed to be treatable in the simplest manner. In the current section we intend to apply the considerations above to other sets of Lorentz-violating parameters in the nonminimal fermion sector, which have richer characteristics and may, therefore, lead to additional complications. All families of coefficients plus their most important properties are listed and summarized in Tab. I in [15].

A. *CPT*-even vector coefficients

The first choice is the family of *CPT*-even vector coefficients \hat{c}^μ , which can be decomposed into a sum of coefficients of even operator dimension:

$$\hat{c}^\mu \equiv \hat{c}^{\mu\alpha_1} p_{\alpha_1} = \sum_{\substack{d \text{ even} \\ d \geq 4}}^{\infty} \hat{c}^{(d)\mu}, \quad \hat{c}^{(d)\mu} \equiv c^{(d)\mu\alpha_1 \dots \alpha_{(d-3)}} p_{\nu} p_{\alpha_1} \dots p_{\alpha_{(d-3)}}, \quad \hat{\mathcal{Q}} = \hat{c}^\mu \gamma_\mu. \quad (6.1)$$

The coefficients of the minimal dimension-4 operator have two indices where the second is contracted with the four-momentum: $\hat{c}^{(4)\mu} = c^{(4)\mu\alpha_1} p_{\alpha_1}$. Considering the dimension-6 contribution we deal with $\hat{c}^{(6)\mu} = c^{(6)\mu\alpha_1\alpha_2\alpha_3} p_{\alpha_1} p_{\alpha_2} p_{\alpha_3}$, which is made up of the 64 coefficients $c^{(6)\mu\alpha_1\alpha_2\alpha_3}$. Since \hat{c}^μ has mass dimension 1, the restricted family of coefficients $c^{(6)\mu\alpha_1\alpha_2\alpha_3}$ has dimension -2 .

1. Modified fermion dispersion relations

In the current section the modified dispersion relations will be determined. Equation (35) in [15] gives the quantities $\hat{\mathcal{S}}_\pm = -m_\psi$, $\hat{\mathcal{V}}_\pm^\mu = p^\mu + \hat{c}^\mu$, and $\hat{\mathcal{T}}_\pm^{\mu\nu} = 0$ that are used in Eq. (39) in the latter reference to obtain the following off-shell dispersion relation:

$$(p + \hat{c})^2 - m_\psi^2 = 0. \quad (6.2)$$

From this polynomial the positive- and negative-energy eigenvalues can be deduced. In what follows, the positive dispersion laws will be given. The temporal sector is characterized by the set of coefficients $c^{(6)\mu\nu 00}$. This is the most complicated sector to handle since it involves two additional time-derivatives in configuration space. The modified dispersion relations for the whole temporal sector are involved, which is why the dispersion laws for certain subsets are given. For a theory with only a nonvanishing $c^{(6)0000}$ the modified dispersion law reads as follows:

$$\tilde{E}_\psi^{(\text{temp},1)} = \frac{1}{\sqrt{6}} \sqrt{\frac{2^{4/3}}{\hat{C}} + \frac{2^{2/3}}{\hat{c}_1^2} \hat{C} - \frac{4}{\hat{c}_1}}, \quad (6.3a)$$

$$\hat{C} = \left\{ \hat{c}_1^3 [2 + 27(\mathbf{p}^2 + m_\psi^2) \hat{c}_1] + 3\sqrt{3} \sqrt{(\mathbf{p}^2 + m_\psi^2) \hat{c}_1^7 [4 + 27(\mathbf{p}^2 + m_\psi^2) \hat{c}_1]} \right\}^{1/3}, \quad (6.3b)$$

$$\hat{c}_1 = c^{(6)0000}. \quad (6.3c)$$

Besides this perturbed dispersion law there are two further dispersion relations, which are spurious. Therefore they will not be stated here. The occurrence of fractional powers other than square roots in Eq. (6.3) traces back to six powers of p_0 in Eq. (6.2).

For the set of coefficients $c^{(6)0i00}$ with $i = 1 \dots 3$ and other parameters vanishing the dispersion relations are

$$\tilde{E}_{\psi;1,2}^{(\text{temp},2)} = \frac{1 \mp \sqrt{1 - 4\sqrt{\mathbf{p}^2 + m_\psi^2} \hat{c}_2}}{2\hat{c}_2}, \quad (6.4a)$$

$$\hat{c}_2 = c^{(6)0i00} p^i. \quad (6.4b)$$

The third set of coefficients, which shall be considered for the temporal sector, is $c^{(6)ij00}$ with the spatial indices i and j leading to the following dispersion relations:

$$\tilde{E}_\psi^{(\text{temp},3)} = \sqrt{\frac{1 + \hat{c}_3 \mp \sqrt{(1 + \hat{c}_3)^2 - 4(\mathbf{p}^2 + m_\psi^2)\hat{c}_4}}{2\hat{c}_4}}, \quad (6.5a)$$

$$\hat{c}_3 = 2c^{(6)ij00} p^i p^j, \quad \hat{c}_4 = c^{(6)ij00} c^{(6)ik00} p^j p^k. \quad (6.5b)$$

The first of Eqs. (6.4) and (6.5) are again perturbed ones and the second are spurious. The double square root structure is specific for the dispersion relations of the temporal sector as long as the polynomial in Eq. (6.2) is of degree four. The spurious dispersion laws can again be removed at first order in Lorentz violation. For the first of the two cases considered, in Eq. (6.2) $c^{(6)0000} p_0^2$ has to be replaced by $c^{(6)0000}(\mathbf{p}^2 + m_\psi^2)$, for the second $c^{(6)0i00} p_0^2$ by $c^{(6)0i00}(\mathbf{p}^2 + m_\psi^2)$, and for the third $c^{(6)ij00} p_0^2$ by $c^{(6)ij00}(\mathbf{p}^2 + m_\psi^2)$. One then obtains

$$\tilde{E}_{\psi'}^{(\text{temp},1)} = \frac{\sqrt{\mathbf{p}^2 + m_\psi^2}}{\left| 1 + (\mathbf{p}^2 + m_\psi^2) \hat{c}_1 \right|}, \quad (6.6a)$$

$$\tilde{E}_{\psi'}^{(\text{temp},2)} = \sqrt{\mathbf{p}^2 + m_\psi^2} + \hat{c}_2(\mathbf{p}^2 + m_\psi^2), \quad (6.6b)$$

and

$$\tilde{E}_{\psi'}^{(\text{temp},3)} = \sqrt{(\mathbf{p}^2 + m_\psi^2) \left[1 - \hat{c}_3 + (\mathbf{p}^2 + m_\psi^2) \hat{c}_4 \right]}, \quad (6.6c)$$

respectively. These are perturbed dispersion laws that coincide with the original perturbed ones (the first of Eqs. (6.3), (6.4), and (6.5), respectively) at first order in Lorentz violation. The spurious versions are removed by this procedure.

The mixed sector is defined by the family of coefficients $c^{(6)\mu\nu 0i}$, $c^{(6)\mu\nu i0}$ where μ, ν are Lorentz indices and i a spatial index. Hence there appears one additional time derivative in configuration space in combination with these coefficients. The modified dispersion relation associated with the whole parameter set is involved, which is why certain subsets are considered. For nonvanishing $c^{(6)00i0}$ and $c^{(6)000i}$, i.e., with the first two Lorentz indices set to zero one obtains

$$\tilde{E}_{\psi;1,2}^{(\text{mixed},1)} = \frac{1 \mp \sqrt{1 - 4\sqrt{\mathbf{p}^2 + m_\psi^2} \hat{c}_5}}{2\hat{c}_5}, \quad (6.7a)$$

$$\hat{c}_5 = (c^{(6)00i0} + c^{(6)000i}) p^i. \quad (6.7b)$$

For the latter coefficients both a perturbed and a spurious dispersion law appear again. For $c^{(6)ijk0}$ and $c^{(6)ij0k}$, i.e., with the first two indices restricted to spatial values the modified dispersion law is given by

$$\tilde{E}_{\psi}^{(\text{mixed},2)} = \frac{\hat{c}_6 + \sqrt{\hat{c}_6^2 + (\mathbf{p}^2 + m_{\psi}^2)(1 - \hat{c}_7)}}{1 - \hat{c}_7}, \quad (6.8a)$$

$$\hat{c}_6 = (c^{(6)ijk0} + c^{(6)ij0k})p^i p^j p^k, \quad (6.8b)$$

$$\hat{c}_7 = (c^{(6)ijk0} + c^{(6)ij0k})(c^{(6)ilm0} + c^{(6)il0m})p^j p^k p^l p^m. \quad (6.8c)$$

For this case there is only a perturbed dispersion relation, but not a spurious one.

For one of the first two indices set to zero and the remaining restricted to spatial values the positive-energy dispersion laws read

$$\tilde{E}_{\psi}^{(\text{mixed},3)} = \sqrt{\frac{\pm [(1 + \hat{c}_8)^2 + 2\hat{c}_9] - \sqrt{[(1 + \hat{c}_8)^2 + 2\hat{c}_9]^2 - 4(\mathbf{p}^2 + m_{\psi}^2)\hat{c}_{10}}}{2\hat{c}_{10}}}, \quad (6.9a)$$

$$\hat{c}_8 = (c^{(6)0i0j} + c^{(6)0ij0})p^i p^j, \quad (6.9b)$$

$$\hat{c}_9 = (c^{(6)i0j0} + c^{(6)i00j})p^i p^j, \quad (6.9c)$$

$$\hat{c}_{10} = (c^{(6)i00j} + c^{(6)i0j0})(c^{(6)i00k} + c^{(6)i0k0})p^j p^k. \quad (6.9d)$$

The spurious solutions in Eqs. (6.7), (6.9) can be removed by the replacements

$$\{c^{(6)00i0}, c^{(6)000i}\}p_0^2 \mapsto \{c^{(6)00i0}, c^{(6)000i}\}(\mathbf{p}^2 + m_{\psi}^2), \quad (6.10a)$$

$$\{c^{(6)0i0j}, c^{(6)0ij0}, c^{(6)i0j0}, c^{(6)i00j}\}p_0 \mapsto \{c^{(6)0i0j}, c^{(6)0ij0}, c^{(6)i0j0}, c^{(6)i00j}\}\sqrt{\mathbf{p}^2 + m_{\psi}^2}, \quad (6.10b)$$

in the off-shell dispersion relation of Eq. (6.2). This leads to the following perturbed dispersion laws where the spurious versions are removed:

$$\tilde{E}_{\psi'}^{(\text{mixed},1)} = \sqrt{\frac{\mathbf{p}^2 + m_{\psi}^2}{1 + \hat{c}_5 \left[(\mathbf{p}^2 + m_{\psi}^2)\hat{c}_5 - 2\sqrt{\mathbf{p}^2 + m_{\psi}^2} \right]}}, \quad (6.11a)$$

$$\tilde{E}_{\psi'}^{(\text{mixed},3)} = \frac{\sqrt{(\mathbf{p}^2 + m_{\psi}^2) \left\{ (\hat{c}_8 + \hat{c}_9)^2 + (1 - \hat{c}_8^2) \left[1 - (\mathbf{p}^2 + m_{\psi}^2)\hat{c}_{10} \right] \right\}} - (\hat{c}_8 + \hat{c}_9)\sqrt{\mathbf{p}^2 + m_{\psi}^2}}{1 - (\mathbf{p}^2 + m_{\psi}^2)\hat{c}_{10}}. \quad (6.11b)$$

Finally, the spatial sector is characterized by the coefficients $c^{(6)\mu\nu ij}$ with the Lorentz indices μ, ν and the spatial indices i and j . Due to the complexity of the general case we restrict this sector to the set of coefficients $c^{(6)\mu ij k}$ with only one Lorentz index μ and three spatial indices i, j , and k .

The following dispersion relation is then associated with these coefficients:

$$\tilde{E}_{\psi}^{(\text{spatial})} = \sqrt{\mathbf{p}^2 + m_{\psi}^2 + \hat{c}_{12} + \hat{c}_{13}}, \quad (6.12a)$$

$$\hat{c}_{12} = c^{(6)ijkl} c^{(6)imno} p^j p^k p^l p^m p^n p^o - 2c^{(6)ijkl} p^i p^j p^k p^l, \quad (6.12b)$$

$$\hat{c}_{13} = c^{(6)0ijk} p^i p^j p^k. \quad (6.12c)$$

Note that at least for some coefficients of the mixed and the spatial sector there are no spurious dispersion laws but only perturbed ones.

The dispersion laws given in the current section correspond to positive energies $\tilde{E}_{\psi}^{(>)} = \tilde{E}_{\psi}^{(>)}(\mathbf{p}, c^{(6)\mu\alpha_1\alpha_2\alpha_3})$. The relation between the positive-energy and the negative-energy solutions $\tilde{E}_{\psi}^{(<)}$ is $\tilde{E}_{\psi}^{(<)}(\mathbf{p}, c^{(6)\mu\alpha_1\alpha_2\alpha_3}) = -\tilde{E}_{\psi}^{(<)}(-\mathbf{p}, c^{(6)\mu\alpha_1\alpha_2\alpha_3})$. This means that both are related by reversing the sign of the four-momentum p^{μ} where the Lorentz-violating coefficients $c^{(6)\mu\alpha_1\alpha_2\alpha_3}$ remain untouched. Also in this case the Feynman-Stückelberg interpretation tells us that a negative-energy particle with four-momentum $(p^{\mu}) = (-p^0, -\mathbf{p})^T$ can be considered as a positive-energy antiparticle with $(p^{\mu}) = (p^0, \mathbf{p})^T$. Hence, given the particle energies $\tilde{E}_{\psi}^{(>)}(\mathbf{p}, c^{(6)\mu\alpha_1\alpha_2\alpha_3})$, after reinterpreting $p^0 = \tilde{E}_{\psi}^{(<)}(\mathbf{p}, c^{(6)\mu\alpha_1\alpha_2\alpha_3})$ with $p^{\mu} \mapsto -p^{\mu}$ the corresponding antiparticle energies result in $p^0 = \tilde{E}_{\psi}^{(>)}(\mathbf{p}, c^{(6)\mu\alpha_1\alpha_2\alpha_3})$. Then the particle and antiparticle dispersion laws are equal.

The minimal coefficients $c^{(4)\mu\alpha_1}$, which are linked to the dimension-4 operator, are both *CPT*-even and *C*-even (see [13] for the transformation properties of the various Lorentz-violating coefficients with respect to *C*, *P*, and *T*). For this reason the $c^{(4)\mu\alpha_1}$ in the positive-energy solutions do not come with a different sign in comparison to the negative-energy solutions [26]. The same holds for the coefficients $c^{(6)\mu\alpha_1\alpha_2\alpha_3}$, which substantiates the computed results.

However, caution is required when talking about \hat{c}^{μ} , which includes additional, contracted four-derivatives in configuration space. For example, the dimension-4 coefficients are contracted with one four-derivative ∂_{α_1} . A four-derivative transforms odd under *CPT*, whereby $c^{(4)\mu\alpha_1}\partial_{\alpha_1}$ is *CPT*-odd as well. Hence based on the *CPT*-handedness of the minimal coefficients the transformation properties of the coefficients contracted with additional four-derivatives depends on the number of these derivatives. Therefore the nonminimal dimension-6 coefficients contracted with three derivatives, $c^{(6)\mu\alpha_1\alpha_2\alpha_3}\partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3}$, transforms as a *CPT*-odd object. Similar arguments are valid in momentum space. This is why for the antiparticle energies of Eq. (65) in [15] the sign in the second term is different from the sign of the particle energies of Eq. (61).

The expansions of Eqs. (6.6a) – (6.12) at first order Lorentz violation agree with the upper 2×2 block of Eq. (59) in [15] and the results for antiparticles agree with the reinterpreted lower 2×2 block of the latter equation.

2. Effective coefficients

Certain coefficients in the fermion sector are related, e.g., \hat{m} and \hat{c}^{μ} [15]. For example, expanding the dispersion relation of Eq. (6.3) for the temporal sector of \hat{c}^{μ} the following result is obtained at first order in the single nonzero Lorentz-violating coefficient:

$$\tilde{E}_{\psi; \hat{c}^0}^{(\text{temp})} = E_{\psi} - c^{(6)0000} E_{\psi}^3 = E_{\psi} - \hat{c}^0, \quad (6.13)$$

with the standard fermion energy E_ψ . Compare this result to the first-order expansion of the dispersion relation of Eq. (3.2b),

$$\tilde{E}_{\psi;\hat{m}}^{(\text{temp})} = E_\psi + m_\psi m^{(5)00} E_\psi = E_\psi + \frac{m_\psi}{E_\psi} \hat{m}, \quad (6.14)$$

which is valid for the temporal sector of \hat{m} . They have a similar structure, i.e., the respective Lorentz-violating coefficients may be related to each other. For this reason an effective coefficient can be introduced that incorporates both the \hat{m} and \hat{c}^μ coefficients. Since \hat{c}^μ transforms as a Lorentz vector and \hat{m} as a Lorentz scalar, the following *ansatz* is proposed for the effective coefficients:

$$\hat{c}_{\text{eff}}^\mu = \alpha \hat{c}^\mu + \beta p^\mu \hat{m}, \quad (6.15)$$

where the four-momentum p^μ is used to provide \hat{m} a vector structure. Now the parameters α and $\beta \in \mathbb{R}$ have to be determined. By contracting the *ansatz* above with $-p_\mu/E_\psi$ and setting $\hat{c}^i = 0$ (for $i = 1 \dots 3$) we try to reproduce the first-order terms in dispersion laws:

$$-\frac{1}{E_\psi} p_\mu \hat{c}_{\text{eff}}^\mu = -\alpha \hat{c}^0 - \beta \frac{m_\psi^2}{E_\psi} \hat{m}. \quad (6.16)$$

Comparing this with Eqs. (6.13) and (6.14), respectively, delivers $\alpha = 1$ and $\beta = -1/m_\psi$. Hence the effective coefficients would be given by

$$\hat{c}_{\text{eff}}^\mu = \hat{c}^\mu - \frac{1}{m_\psi} p^\mu \hat{m}. \quad (6.17)$$

This is in accordance with the second equation of Eq. (26) in [15]. Now let us look at the zeroth component:

$$\hat{c}_{\text{eff}}^0 \equiv \hat{c}^0 - \frac{E_\psi}{m_\psi} \hat{m}, \quad c_{\text{eff}}^{(6)0000} = c^{(6)0000} - \frac{1}{m_\psi} m^{(5)00}. \quad (6.18)$$

The latter result coincides with the second equation of Eq. (27) in [15] for $d = 6$. The next step is to compare the expansions of the dispersion relations for the spatial sector of Eqs. (3.4b) and (6.12):

$$\tilde{E}_{\psi;\hat{m}}^{(\text{spat})} = E_\psi + \frac{m_\psi}{E_\psi} m^{(5)kl} p^k p^l = E_\psi + \frac{m_\psi}{E_\psi} \hat{m}, \quad (6.19a)$$

$$\tilde{E}_{\psi;\hat{c}^i}^{(\text{spat})} = E_\psi - \frac{1}{E_\psi} c^{(6)ijkl} p^i p^j p^k p^l = E_\psi + \frac{1}{E_\psi} \hat{c}^i p^i. \quad (6.19b)$$

Repeating the procedure above, i.e., contracting the *ansatz* for the effective coefficient with $-p_\mu/E_\psi$ and setting $\hat{c}^0 = 0$ result in

$$-\frac{1}{E_\psi} p_\mu \hat{c}_{\text{eff}}^\mu = \frac{1}{E_\psi} \alpha p^i \hat{c}^i - \beta \frac{m_\psi^2}{E_\psi} \hat{m}, \quad (6.20)$$

from which $\alpha = 1$ and $\beta = -1/m_\psi$ follows when it is compared to Eqs. (6.19a) and (6.19b). Hence the *ansatz* is consistent for both sectors. Considering the i -th component of \hat{c}_{eff}^μ and multiplying it with p^i leads to

$$\hat{c}_{\text{eff}}^i p^i \equiv \hat{c}^i p^i - \frac{1}{m_\psi} \delta^{ij} p^i p^j \hat{m}, \quad c_{\text{eff}}^{(6)ijkl} = c^{(6)ijkl} + \frac{1}{m_\psi} \delta^{ij} m^{(5)kl}. \quad (6.21)$$

which is again in accordance with the second equation of Eq. (27) in [15] for $d = 6$. Similar deliberations can be done for the other coefficients. This provides is a good cross check for the results obtained.

3. Modified spinors and completeness relations

One the one hand, according to [15] the positive-energy spinors can be written as follows:

$$u^{(\alpha)}(\tilde{E}_\psi^{(>)}, \mathbf{p}) = \frac{1}{\sqrt{N_u^{(\alpha)}}} U^\dagger(\tilde{E}_\psi^{(>)} + \hat{c}^0, m_\psi, \mathbf{p} + \hat{\mathbf{c}}) u^{(\alpha)}(m_\psi, \mathbf{0}), \quad \hat{c}^\mu = \hat{c}^\mu(\tilde{E}_\psi^{(>)}, \mathbf{p}), \quad (6.22)$$

where the $u^{(\alpha)}(m_\psi, \mathbf{0})$ are given in Eq. (4.5b). These spinors are a solution of the modified Dirac equation $(\not{p} + \not{\tilde{\mathcal{E}}} - m_\psi \mathbb{1}_4) u^{(\alpha)}(p) = 0$ with $(p^\mu) = (\tilde{E}_\psi^{(>)}, \mathbf{p})^T$. The modified Dirac equation and the spinor solution show that \hat{c}^μ is tightly connected to the particle four-momentum. On the other hand, the negative-energy spinors are given by:

$$v^{(\alpha)}(\tilde{E}_\psi^{(>)}, \mathbf{p}) = \frac{1}{\sqrt{N_v^{(\alpha)}}} U^\dagger(\tilde{E}_\psi^{(>)} + \hat{c}^0, m_\psi, -\mathbf{p} - \hat{\mathbf{c}}) v^{(\alpha)}(m_\psi, \mathbf{0}), \quad (6.23a)$$

$$\hat{c}^\mu = \hat{c}^\mu(-\tilde{E}_\psi^{(>)}, -\mathbf{p}) = -\hat{c}^\mu(\tilde{E}_\psi^{(>)}, \mathbf{p}), \quad (6.23b)$$

with the $v^{(\alpha)}(m_\psi, \mathbf{0})$ of Eq. (4.8b). The property (6.23b) of the \hat{c}^μ coefficients is valid since they contain a combination of three four-momenta. These spinors obey the modified Dirac equation $(\not{p} + \not{\tilde{\mathcal{E}}} - m_\psi \mathbb{1}_4) v^{(\alpha)}(p) = 0$ with $(p^\mu) = (-\tilde{E}_\psi^{(>)}, -\mathbf{p})^T$. The normalizations $N_u^{(\alpha)}$ and $N_v^{(\alpha)}$ of the positive- and negative-energy spinors are chosen such that

$$\bar{u}^{(\alpha)}(p) u^{(\beta)}(p) = 2m_\psi \delta^{\alpha\beta}, \quad \bar{v}^{(\alpha)}(p) v^{(\beta)}(p) = -2m_\psi \delta^{\alpha\beta}. \quad (6.24)$$

The modified spinor completeness relations are given by:

$$\sum_{\alpha=1,2} u^{(\alpha)}(p) \bar{u}^{(\alpha)}(p) = \not{p} + \not{\tilde{\mathcal{E}}} + m_\psi \mathbb{1}_4, \quad (6.25a)$$

$$\sum_{\alpha=1,2} v^{(\alpha)}(p) \bar{v}^{(\alpha)}(p) = \not{p} + \not{\tilde{\mathcal{E}}} - m_\psi \mathbb{1}_4, \quad (6.25b)$$

where $(p^\mu) = (\tilde{E}_\psi^{(>)}, \mathbf{p})^T$ on the right-hand sides of the latter two relations. All these relations can be shown analogously to the relations for the coefficient \hat{m} , cf. Appx. A 1, A 2, which again indicates that \hat{m} and \hat{c}^μ are related. With the propagator

$$S(p) = \frac{i}{(p + \hat{c})^2 - m_\psi^2} (\not{p} + \not{\tilde{\mathcal{E}}} + m_\psi \mathbb{1}_4), \quad (6.26)$$

the proof of the optical theorem for the process considered in the last chapter can be done completely analogously. Note that in the forward scattering amplitude for the positron given by Eq. (5.6) the sign of the four-momentum vector has to be reversed where, as a result of this, the sign of \hat{c} changes as well. This leads to the second completeness relation of Eq. (6.25). Furthermore for the check of the optical theorem for sets of coefficients with spurious dispersion relations their replacements, which are valid at first order Lorentz violation, have to be used (e.g., Eqs. (6.6a) – (6.6c) for the temporal sector and Eqs. (6.11a), (6.11b) for the mixed sector of the coefficient \hat{c}^μ).

B. *CPT*-odd pseudoscalar coefficients

Only for the Lorentz-violating coefficients previously considered the diagonalization of the Dirac operator $\not{p} - m_\psi \mathbb{1}_4 + \hat{\mathcal{Q}}$ can be performed with the matrix U given by Eq. (4.3). For all remaining cases diagonalization is more involved and an analogue of U valid at all orders in Lorentz violation is not at hand so far. However it was shown that in these cases the Dirac operator can be block-diagonalized at least at first order Lorentz violation by the following matrix [15]:

$$U^{(1)} = \left(\mathbb{1}_4 + \frac{1}{4E_\psi} [\gamma_5, R] \right) VW, \quad R = VW\gamma_0 \hat{\mathcal{Q}} W^\dagger V^\dagger, \quad W = W(E_\psi, m_\psi, \mathbf{p}), \quad (6.27)$$

with V and W of Eq. (4.3). Here the index of U indicates that this result is valid at first order in Lorentz violation. Note that the expression involves the standard fermion dispersion relation E_ψ .

As a next example the family of *CPT*-odd pseudoscalar coefficients $\hat{f} \equiv \hat{f}^{\alpha_1} p_{\alpha_1}$ will be considered. They can be written as a series of coefficients with even operator dimension:

$$\hat{f} \equiv \hat{f}^{\alpha_1} p_{\alpha_1} = \sum_{\substack{d \text{ even} \\ d \geq 4}}^{\infty} \hat{f}^{(d)}, \quad \hat{f}^{(d)} \equiv f^{(d)\alpha_1 \dots \alpha_{(d-3)}} p_{\alpha_1} \dots p_{\alpha_{(d-3)}}, \quad \hat{\mathcal{Q}} = i\hat{f}\gamma^5. \quad (6.28)$$

The minimal extension comprises the dimension-4 operator with $\hat{f}^{(4)} = f^{(4)\alpha_1} p_{\alpha_1}$ where the corresponding coefficients $f^{(4)\alpha_1}$ are contracted with the four-momentum. In this particular section the dimension-6 operator will be considered, i.e., $\hat{f}^{(6)} = f^{(6)\alpha_1 \alpha_2 \alpha_3} p_{\alpha_1} p_{\alpha_2} p_{\alpha_3}$. The set \hat{f} has mass dimension 1 and the 20 coefficients $f^{(6)\alpha_1 \alpha_2 \alpha_3}$ have mass dimension -2 . We again split this set of parameters in a temporal, mixed, and a spatial sector. The temporal sector consists of the single coefficient $f^{(6)\alpha_1 00}$, the mixed sector is made up of $f^{(6)\alpha_1 0i}$, $f^{(6)\alpha_1 i0}$ with the spatial index i and the spatial sector comprises $f^{(6)\alpha_1 ij}$ with $i, j = 1 \dots 3$.

1. Modified dispersion relations

The modified positive-energy dispersion laws $\tilde{E}_\psi^{(>)}$ follow from the condition $\det(\not{p} - m_\psi \mathbb{1}_4 + \hat{\mathcal{Q}}) = 0$ with $\hat{\mathcal{Q}} = i\hat{f}\gamma^5$. An alternative is to use Eq. (39) of [15] with $\hat{\mathcal{S}}_\pm = -m_\psi \pm i\hat{f}$, $\hat{\mathcal{V}}_\pm^\mu = p^\mu$, and $\hat{\mathcal{T}}_\pm^{\mu\nu} = 0$. For the temporal sector with the single nonvanishing coefficient $f^{(6)000}$ there are two distinct positive energies. The first reads

$$\tilde{E}_{\psi;1}^{(\text{temp},1)} = \frac{1}{3^{1/4}} \sqrt{\frac{2}{|\hat{f}_1|}} \sin\left(\frac{u}{3} + \frac{\pi}{6}\right), \quad (6.29a)$$

$$u = -\arctan\left(\frac{\sqrt{12 - 81(\mathbf{p}^2 + m_\psi^2)^2 (\hat{f}_1)^2}}{9(\mathbf{p}^2 + m_\psi^2)|\hat{f}_1|}\right), \quad (6.29b)$$

$$\hat{f}_1 = f^{(6)000}, \quad (6.29c)$$

and the second is given by

$$\tilde{E}_{\psi;2}^{(\text{temp},1)} = \frac{1}{6^{1/3}} \sqrt{\frac{2 \cdot 3^{1/3}}{v^{1/3}} + \frac{(2v)^{1/3}}{(\hat{f}_1)^2}}, \quad (6.30a)$$

$$v = \sqrt{81(\mathbf{p}^2 + m_\psi^2)^2(\hat{f}_1)^8 - 12(\hat{f}_1)^6 - 9(\mathbf{p}^2 + m_\psi^2)(\hat{f}_1)^4}. \quad (6.30b)$$

These involve trigonometric functions and fractional powers other than square roots, which are rather unusual functions to appear in the context of modified dispersion laws. The reason for their occurrence is that the modified determinant condition is a polynomial of sixth degree in p_0 .³ The first dispersion law is perturbed and the second is spurious, which becomes evident from the following expansions in the Lorentz-violating coefficient:

$$\tilde{E}_{\psi;1}^{(\text{temp},1)} = \sqrt{\mathbf{p}^2 + m_\psi^2} \left[1 + \frac{1}{2}(\hat{f}_1)^2(\mathbf{p}^2 + m_\psi^2) \right] + \mathcal{O}[(\hat{f}_1)^4], \quad (6.31a)$$

$$\tilde{E}_{\psi;2}^{(\text{temp},1)} = \frac{1}{\sqrt{\hat{f}_1}} - \frac{1}{4}\sqrt{\hat{f}_1}(\mathbf{p}^2 + m_\psi^2) + \mathcal{O}[(\hat{f}_1)^{3/2}]. \quad (6.31b)$$

Considering the temporal sector with the three coefficients $f^{(6)i00}$ for $i = 1 \dots 3$ and all remaining set to zero the positive-energy dispersion laws are

$$\tilde{E}_{\psi;1,2}^{(\text{temp},2)} = \frac{\sqrt{1 \mp \sqrt{1 - 4(\hat{f}_2)^2(\mathbf{p}^2 + m_\psi^2)}}}{\sqrt{2}|\hat{f}_2|}, \quad (6.32a)$$

$$\hat{f}_2 = f^{(6)i00}p^i. \quad (6.32b)$$

Here the first is perturbed and the second spurious. The double square root structure appears again since the determinant condition is a polynomial of fourth degree in p_0 . The spurious dispersion laws of Eqs. (6.30), (6.32) can be removed at first order Lorentz violation by the replacement $f^{(6)000}p_0^2 \mapsto f^{(6)000}(\mathbf{p}^2 + m_\psi^2)$ in the determinant condition. This leads to

$$\tilde{E}_{\psi'}^{(\text{temp},1)} = \sqrt{\frac{\mathbf{p}^2 + m_\psi^2}{1 - (\mathbf{p}^2 + m_\psi^2)^2(\hat{f}_1)^2}} \quad (6.33a)$$

$$\tilde{E}_{\psi'}^{(\text{temp},2)} = \sqrt{(\mathbf{p}^2 + m_\psi^2) \left[1 + (\mathbf{p}^2 + m_\psi^2)(\hat{f}_2)^2 \right]}. \quad (6.33b)$$

For the mixed sector first of all, the coefficients are considered with the first Lorentz index equal to zero. The positive-energy dispersion laws read

$$\tilde{E}_{\psi;1,2}^{(\text{mixed},1)} = \frac{\sqrt{1 \mp \sqrt{1 - 4(\hat{f}_3)^2(\mathbf{p}^2 + m_\psi^2)}}}{\sqrt{2}|\hat{f}_3|}, \quad (6.34a)$$

$$\hat{f}_3 = (f^{(6)0i0} + f^{(6)00i})p^i. \quad (6.34b)$$

³ Dispersion relations involving trigonometric functions and fractional powers different from mere square roots also appear in the mixed sector of the dimension-6 coefficients $\kappa_{\text{tr}}^{\mu\nu}$ in the nonminimal SME photon sector [24].

Here the first is perturbed and the second is spurious. Equations (6.32) and (6.34) can, in principle, be merged to a single dispersion relation dependent on the combination of the two sets of coefficients. To remove the spurious dispersion law in Eq. (6.34) the replacement $f^{(6)i00}p_0 \mapsto f^{(6)i00}\sqrt{\mathbf{p}^2 + m_\psi^2}$ in the determinant condition has to be performed, which leads to

$$\tilde{E}_{\psi'}^{(\text{mixed},1)} = \sqrt{\frac{\mathbf{p}^2 + m_\psi^2}{1 - (\mathbf{p}^2 + m_\psi^2)(\hat{f}_3)^2}}. \quad (6.35)$$

Note the similarities between Eqs. (6.33a) and (6.35).

Second, only the coefficients where the first Lorentz index has a spatial value are taken into account. For this particular set one then obtains

$$\tilde{E}_\psi^{(\text{mixed},2)} = \sqrt{\frac{\mathbf{p}^2 + m_\psi^2}{1 - (\hat{f}_4)^2}}, \quad (6.36a)$$

$$\hat{f}_4 = (f^{(6)ij0} + f^{(6)i0j})p^i p^j. \quad (6.36b)$$

Finally, for the spatial sector it follows that

$$\tilde{E}_\psi^{(\text{spatial})} = \frac{-\hat{f}_5 \hat{f}_6 + \sqrt{[\mathbf{p}^2 + m_\psi^2 + (\hat{f}_6)^2][1 - (\hat{f}_5)^2] + (\hat{f}_5 \hat{f}_6)^2}}{1 - (\hat{f}_5)^2}, \quad (6.37a)$$

$$\hat{f}_5 = f^{(6)0ij}p^i p^j, \quad \hat{f}_6 = f^{(6)ijk}p^i p^j p^k. \quad (6.37b)$$

For at least some coefficients of the spatial and the mixed sector there are only perturbed but no spurious dispersion laws. The negative-energy solutions are related to be positive-energy solutions by $\tilde{E}_\psi^{(>)}(\mathbf{p}, f^{(6)\alpha_1\alpha_2\alpha_3}) = -\tilde{E}_\psi^{(<)}(-\mathbf{p}, -f^{(6)\alpha_1\alpha_2\alpha_3})$. Note that contrary to the cases with the \hat{m} and the \hat{c}^μ coefficients, the $f^{(6)\alpha_1\alpha_2\alpha_3}$ come with a minus sign on the right-hand side of the latter relation. This indicates their *CPT*-odd nature and the same behavior is observed for the minimal, *CPT*-odd SME coefficients in the fermion sector [26]. So if $p^0 = \tilde{E}_\psi^{(<)}(\mathbf{p}, f^{(6)\alpha_1\alpha_2\alpha_3})$ is reinterpreted with the transformation $p^\mu \mapsto -p^\mu$, the antiparticle energies will be $p^0 = \tilde{E}_\psi^{(>)}(\mathbf{p}, -f^{(6)\alpha_1\alpha_2\alpha_3})$. The latter differ from the corresponding particle energies $\tilde{E}_\psi^{(>)}(\mathbf{p}, f^{(6)\alpha_1\alpha_2\alpha_3})$ due to the minus sign associated with the $f^{(6)\alpha_1\alpha_2\alpha_3}$. This is expected for a theory violating *CPT*.

Furthermore, in contrast to the cases of the coefficients \hat{m} and \hat{c} the Lorentz-violating coefficients \hat{f} only appear at quadratic and higher (even) orders in the dispersion relations. In [15] it was stated that these coefficients can be removed from the physical observables by a field redefinition at first order Lorentz violation, which reflects the results obtained here. Nevertheless it is reasonable to investigate the properties of this family of coefficients, because they comprise the simplest set of higher-dimensional *CPT*-odd coefficients. An alternative would be the scalar coefficients \hat{e} in Tab. I of [15]. However their properties are expected to be similar to the properties of \hat{m} , which were already considered.

C. Connection to the coefficients \hat{c}^μ

In [36] it was shown that at leading order in the minimal Lorentz-violating coefficients $f^{(4)\alpha_1}$ all Lorentz-violating modifications in observables cannot be distinguished from those of the $c^{(4)\mu\alpha_1}$

coefficients: $c^{(4)\mu\alpha_1} = -f^{(4)\mu}f^{(4)\alpha_1}/2$. In this context by leading order it is referred to the first nonvanishing terms in expansions with respect to the Lorentz-violating coefficients. This means first order for the coefficients $c^{(4)\mu\alpha_1}$ and second order for the coefficients $f^{(4)\alpha_1}$.

It can be investigated that an analogue relation holds for the nonminimal coefficients $\hat{c}^{(6)\mu}$ and $\hat{f}^{(6)}$ considered in this article as well. For example, the leading-order expansions of Eqs. (6.3), (6.29),

$$\tilde{E}_{\psi,\hat{c}^\mu} = E_\psi - c^{(6)0000}E_\psi^3, \quad \tilde{E}_{\psi,\hat{f}} = E_\psi + \frac{1}{2}(f^{(6)000})^2E_\psi^5, \quad (6.38)$$

correspond to each other for $c^{(6)0000}E_\psi^2 = -(f^{(6)000})^2E_\psi^4/2$. Second, comparing similar expansions of Eqs. (6.5), (6.32) results in

$$\tilde{E}_{\psi,\hat{c}^\mu} = E_\psi - c^{(6)ij00}p^ip^jE_\psi, \quad \tilde{E}_{\psi,\hat{f}} = E_\psi + \frac{1}{2}(f^{(6)i00}p^i)(f^{(6)j00}p^j)E_\psi^3, \quad (6.39)$$

which are the same for $c^{(6)ij00}E_\psi^2 = -f^{(6)i00}f^{(6)j00}E_\psi^4/2$. Finally, the leading-order expansions of Eqs. (6.12), (6.37) read as follows:

$$\tilde{E}_{\psi,\hat{c}^\mu} = E_\psi - \frac{c^{(6)ijkl}p^ip^jp^kp^l}{E_\psi}, \quad \tilde{E}_{\psi,\hat{f}} = E_\psi + \frac{(f^{(6)ijk}p^ip^jp^k)(f^{(6)lmn}p^lp^mp^n)}{2E_\psi}, \quad (6.40)$$

where the coefficients $f^{(6)0ij}$ are set to zero in the second expansion. Both expressions are evidently equal for $c^{(6)ijkl}p^kp^l = -f^{(6)ikl}p^kp^lf^{(6)jmn}p^mp^n/2$. Hence I conjecture that the following identity even holds for the nonminimal coefficients at leading order:

$$\hat{c}^{(6)\mu\alpha_1} = -\frac{1}{2}\hat{f}^{(6)\mu}\hat{f}^{(6)\alpha_1}, \quad (6.41)$$

So the nonminimal \hat{f} coefficients can most probably also be absorbed in the nonminimal \hat{c}^μ coefficients at this order. This is another motivation to consider the \hat{f} coefficients in the current paper. Computations can possibly be done with less effort for the \hat{f} rather than for the \hat{c}^μ coefficients, since the first have a simpler momentum structure.

There is a further interesting fact on the minimal coefficients $f^{\alpha_1}p_{\alpha_1}$ in the context of classical Lagrangians. In [37] the Lagrangians of a classical point particle obeying the Lorentz-violating kinematics were obtained for certain minimal coefficients. If all coefficients vanish except of the $f^{\alpha_1}p_{\alpha_1}$, the Lagrangian of their Eq. (8) only depends on a quadratic combination of these coefficients. This is in accordance to the structure of the dispersion relations obtained in the previous section where only quadratic powers of the nonminimal coefficients \hat{f} appear.

1. Modified spinors and completeness relations

For the particular case of the pseudoscalar coefficients \hat{f} the Dirac equation cannot only be block-diagonalized with the matrix $U^{(1)}$ of Eq. (6.27) but it can be diagonalized completely. Because of this the positive-energy spinors at first order Lorentz violation can be obtained with the Hermitian conjugate of the matrix $U^{(1)}$. They are given by

$$u^{(\alpha)}(\tilde{E}_\psi^{(>)}, \mathbf{p}) = \frac{1}{\sqrt{N_u^{(\alpha)}}}U^{(1)\dagger}(E_\psi^{(>)}, m_\psi, \mathbf{p})u^{(\alpha)}(m_\psi, \mathbf{0}), \quad (6.42a)$$

$$\hat{\mathcal{Q}} = \hat{\mathcal{Q}}(\tilde{E}_\psi^{(>)}, \mathbf{p}), \quad (p^\mu) = (\tilde{E}_\psi^{(>)}, \mathbf{p})^T. \quad (6.42b)$$

The negative-energy spinors read

$$v^{(\alpha)}(\tilde{E}_\psi^{(>)}, \mathbf{p}) = \frac{1}{\sqrt{N_v^{(\alpha)}}} U^{(1)\dagger}(E_\psi^{(>)}, m_\psi, -\mathbf{p}) v^{(\alpha)}(m_\psi, \mathbf{0}), \quad (6.43a)$$

$$\hat{\mathcal{Q}} = \hat{\mathcal{Q}}(-\tilde{E}_\psi^{(>)}, -\mathbf{p}) = -\hat{\mathcal{Q}}(\tilde{E}_\psi^{(>)}, \mathbf{p}), \quad (p^\mu) = (-\tilde{E}_\psi^{(>)}, -\mathbf{p})^T. \quad (6.43b)$$

The spinor normalizations $N_u^{(\alpha)}$ and $N_v^{(\alpha)}$ are chosen analogously to Eq. (6.24). The explicit expressions for the spinors are obtained in App. A 3. In the latter section of the appendix the following completeness relations are deduced as well:

$$\sum_{\alpha=1,2} u^{(\alpha)}(p) \bar{u}^{(\alpha)}(p) = \not{p} + m_\psi \mathbb{1}_4 + \hat{\mathcal{Q}} + \mathcal{O}(\hat{f}^2), \quad (6.44a)$$

$$\sum_{\alpha=1,2} v^{(\alpha)}(p) \bar{v}^{(\alpha)}(p) = \not{p} - m_\psi \mathbb{1}_4 - \hat{\mathcal{Q}} + \mathcal{O}(\hat{f}^2), \quad (6.44b)$$

where $(p^\mu) = (\tilde{E}_\psi^{(>)}, \mathbf{p})^T$ on the right-hand sides of the latter completeness relations. For the negative-energy spinors both $m_\psi \mathbb{1}_4$ and $\hat{\mathcal{Q}}$ come with a minus sign, which is crucial for the validity of the optical theorem. The propagator is derived as usual by inverting the Dirac operator in momentum space, $S^{-1}(p) = \not{p} - m_\psi \mathbb{1}_4 - \hat{\mathcal{Q}}$, and expressing the result via the Dirac matrices needed. For the case considered the *ansatz*

$$S(p) = a_1 \gamma^0 + a_2 \gamma^1 + a_3 \gamma^2 + a_4 \gamma^3 + a_5 \mathbb{1}_4 + a_6 \gamma^5, \quad (6.45)$$

is sufficient because these are the Dirac matrices that appear in the Dirac operator. Solving the resulting linear system of equations with respect to the coefficients a_i leads to the modified propagator:

$$S(p) = \frac{i}{p^2 - (m_\psi^2 + \hat{f}^2)} (\not{p} + m_\psi \mathbb{1}_4 + \hat{\mathcal{Q}}) = \frac{i}{p^2 - m_\psi^2} (\not{p} + m_\psi \mathbb{1}_4 + \hat{\mathcal{Q}}) + \mathcal{O}(\hat{f}^2). \quad (6.46)$$

With the latter result and the spinor completeness relations of Eq. (6.44) the validity of the optical theorem at tree-level can be demonstrated for both electrons and positrons. The proof works such as for the case of coefficients \hat{m} , cf. Sec. V. Note that with the expressions given the proof can only be done at first order Lorentz violation. The spurious fermion dispersion relations, which may spoil the validity of the optical theorem for the temporal and mixed sector of the coefficients considered, are removed at first order in Lorentz violation according to Sec. VIB 1. For the replacements of Eqs. (6.33a), (6.33b), and (6.35) the proof works similarly.

VII. CONCLUSION AND OUTLOOK

To summarize, in the current article certain properties of quantum field theories that are based on the nonminimal spin-1/2 fermion sector of the Lorentz-violating Standard-Model Extension were examined. For two *CPT*-even and one *CPT*-odd set of Lorentz-violating coefficients the modified fermion dispersion relations, the spinors, completeness relations, and the fermion propagator were obtained. For some subsets of these coefficients spurious dispersion laws emerge that are not a

perturbation of the standard dispersion relation. It was demonstrated that these can be removed at first order Lorentz violation. Furthermore the validity of the optical theorem at tree-level for both fermions and antifermions was proven. For the *CPT*-even families of coefficients the proof is exact in the Lorentz-violating parameters where for the *CPT*-odd case it was performed at first order in Lorentz violation.

In the framework of the analysis performed no issues were found for the set of coefficients considered. Hence these sets seem to result in well-behaved quantum field theories. The spinors, completeness relations, and propagators determined can be used in upcoming particle physics calculations related to phenomenology. Furthermore the methods demonstrated can be applied to investigate the remaining sets of coefficients that were not considered in the paper. Especially the fermion sector of the nonminimal SME is still a *terra incognita* for both experiment [38] and theory.

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Appendix A: Explicit spinors and completeness relations

In the following sections the explicit expressions for the modified Dirac spinors shall be obtained for the various sectors considered. It is convenient to perform the calculations with the matrices and spinors in 2×2 block form at first.

1. *CPT*-even scalar coefficient \widehat{m}

The explicit positive-energy spinors can be obtained directly from Eq. (4.5) by using the Hermitian conjugate of the transformation matrix U given in Eq. (4.3). The latter reads

$$U^\dagger = \begin{pmatrix} \mathbf{n} \cdot \boldsymbol{\sigma} + n^0 \mathbb{1}_2 & \mathbf{n} \cdot \boldsymbol{\sigma} - n^0 \mathbb{1}_2 \\ -\mathbf{n} \cdot \boldsymbol{\sigma} + n^0 \mathbb{1}_2 & \mathbf{n} \cdot \boldsymbol{\sigma} + n^0 \mathbb{1}_2 \end{pmatrix}, \quad (\text{A.1a})$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ with the Pauli matrices σ^1 , σ^2 , and σ^3 of Eq. (4.2b). For convenience the four-vector $(n^\mu) = (n^0, \mathbf{n})$ is introduced with the following components:

$$\mathbf{n} \equiv \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -\frac{\mathbf{p}}{2\sqrt{\widetilde{E}_\psi(\widetilde{E}_\psi + \widetilde{m}_\psi)}}, \quad n^0 \equiv \delta = \frac{\sqrt{\widetilde{E}_\psi(\widetilde{E}_\psi + \widetilde{m}_\psi)}}{2\widetilde{E}_\psi}. \quad (\text{A.1b})$$

With these quantities the positive-energy spinors can be cast in the form

$$u^{(1)}(p) = \frac{1}{\sqrt{N_u^{(1)}}} U^\dagger(\tilde{E}_\psi^{(>)}, \tilde{m}_\psi, \mathbf{p}) \begin{pmatrix} \phi^{(1)} \\ \mathbf{0} \end{pmatrix} = \frac{1}{\sqrt{N_u^{(1)}}} \begin{pmatrix} \gamma + \delta \\ \alpha + i\beta \\ -\gamma + \delta \\ -\alpha - i\beta \end{pmatrix}, \quad (\text{A.2a})$$

$$u^{(2)}(p) = \frac{1}{\sqrt{N_u^{(2)}}} U^\dagger(\tilde{E}_\psi^{(>)}, \tilde{m}_\psi, \mathbf{p}) \begin{pmatrix} \phi^{(2)} \\ \mathbf{0} \end{pmatrix} = \frac{1}{\sqrt{N_u^{(2)}}} \begin{pmatrix} \alpha - i\beta \\ -\gamma + \delta \\ -\alpha + i\beta \\ \gamma + \delta \end{pmatrix}, \quad (\text{A.2b})$$

$$N_u^{(1)} = N_u^{(2)} = \frac{1}{\tilde{m}_\psi} n^2. \quad (\text{A.2c})$$

Now the completeness relation for the positive-energy spinors results in

$$\sum_{\alpha=1,2} u^{(\alpha)}(p) \bar{u}^{(\alpha)}(p) = \frac{\tilde{m}_\psi}{n^2} \begin{pmatrix} M_0 & 0 & M_+ & -M^* \\ 0 & M_0 & -M & M_- \\ M_- & M^* & M_0 & 0 \\ M & M_+ & 0 & M_0 \end{pmatrix}, \quad (\text{A.3a})$$

$$M_0 = n^2, \quad (\text{A.3b})$$

$$M_+ = \alpha^2 + \beta^2 + (\gamma + \delta)^2, \quad M_- = \alpha^2 + \beta^2 + (\gamma - \delta)^2, \quad (\text{A.3c})$$

$$M = -2(\alpha + i\beta)\delta, \quad M^* = -2(\alpha - i\beta)\delta. \quad (\text{A.3d})$$

From the determinant condition it follows that $\tilde{E}_\psi^2 - \tilde{m}_\psi^2 = \mathbf{p}^2$, which can be used to obtain the positive-energy completeness relation of Eq. (4.7).

The negative-energy spinors follow from Eq. (4.8) and they are given by:

$$v^{(1)}(p) = \frac{1}{\sqrt{N_v^{(1)}}} U^\dagger(\tilde{E}_\psi^{(>)}, \tilde{m}_\psi, -\mathbf{p}) \begin{pmatrix} \mathbf{0} \\ \chi^{(1)} \end{pmatrix} = -\frac{1}{\sqrt{N_v^{(1)}}} \begin{pmatrix} \gamma + \delta \\ \alpha + i\beta \\ \gamma - \delta \\ \alpha + i\beta \end{pmatrix}, \quad (\text{A.4a})$$

$$v^{(2)}(p) = \frac{1}{\sqrt{N_v^{(2)}}} U^\dagger(\tilde{E}_\psi^{(>)}, \tilde{m}_\psi, -\mathbf{p}) \begin{pmatrix} \mathbf{0} \\ \chi^{(2)} \end{pmatrix} = -\frac{1}{\sqrt{N_v^{(2)}}} \begin{pmatrix} \alpha - i\beta \\ -\gamma + \delta \\ \alpha - i\beta \\ -(\gamma + \delta) \end{pmatrix}, \quad (\text{A.4b})$$

$$N_v^{(1)} = N_v^{(2)} = \frac{1}{\tilde{m}_\psi} n^2. \quad (\text{A.4c})$$

The completeness relation for the negative-energy spinors is then

$$\sum_{\alpha=1,2} v^{(\alpha)}(p) \bar{v}^{(\alpha)}(p) = \frac{\tilde{m}_\psi}{n^2} \begin{pmatrix} \bar{M}_0 & 0 & M_+ & -M^* \\ 0 & \bar{M}_0 & -M & M_- \\ M_- & M^* & \bar{M}_0 & 0 \\ M & M_+ & 0 & \bar{M}_0 \end{pmatrix}, \quad (\text{A.5a})$$

$$\bar{M}_0 = -n^2, \quad (\text{A.5b})$$

where M_+ , M_- , M , and M^* are given by Eq. (A.3c). With Eq. (A.1b) this leads to the result of Eq. (4.10).

2. *CPT*-even vector coefficient \hat{c}

For this family of coefficients the computations of the previous section can be performed completely analogously with the replacements $\tilde{m}_\psi \mapsto m_\psi$ plus $p^\mu \mapsto p^\mu + \hat{c}^\mu$ for both the positive-energy and the negative-energy spinors (but the momentum components in \hat{c}^μ itself remain untouched, of course). With this knowledge the spinor completeness relations of Eq. (6.25) can be computed. Here it is convenient to use $(\tilde{E}_\psi + \hat{c}^0)^2 - m_\psi^2 = (\mathbf{p} + \hat{\mathbf{c}})^2$, which is obtained from Eq. (6.2).

3. *CPT*-odd pseudoscalar coefficient \hat{f}

In this case the diagonalization matrix U is computed at first order Lorentz violation. It results from Eq. (6.27) and its Hermitian conjugate is explicitly given by:

$$U^{(1)\dagger} = \begin{pmatrix} \mathbf{n} \cdot \boldsymbol{\sigma} + n^0 \mathbb{1}_2 & \mathbf{n} \cdot \boldsymbol{\sigma} - n^0 \mathbb{1}_2 \\ -\mathbf{n}^* \cdot \boldsymbol{\sigma} + (n^0)^* \mathbb{1}_2 & \mathbf{n}^* \cdot \boldsymbol{\sigma} + (n^0)^* \mathbb{1}_2 \end{pmatrix} \quad (\text{A.6a})$$

$$\mathbf{n} \equiv \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -\frac{\mathbf{p}\mathcal{C}}{8E_\psi^{5/2}(E_\psi + m_\psi)^{3/2}}, \quad n^0 \equiv \delta = \frac{\mathcal{C}^*}{8E_\psi^2 \sqrt{E_\psi(E_\psi + m_\psi)}}, \quad (\text{A.6b})$$

$$\mathcal{C} = 4E_\psi^3 + 2i\hat{f}^{(6)}E_\psi m_\psi + i\hat{f}^{(6)}(\mathbf{p}^2 + m_\psi^2) + E_\psi^2(4m_\psi + i\hat{f}^{(6)}), \quad (\text{A.6c})$$

$$\hat{f}^{(6)} = f^{(6)\alpha_1\alpha_2\alpha_3} p_{\alpha_1} p_{\alpha_2} p_{\alpha_3}. \quad (\text{A.6d})$$

Note that the latter formulae involve the standard fermion energy E_ψ instead of the modification \tilde{E}_ψ . The positive-energy spinors then read as

$$u^{(1)}(p) = \frac{1}{\sqrt{N_u^{(1)}}} U^{(1)\dagger}(\tilde{E}_\psi^{(>)}, m_\psi, \mathbf{p}) \begin{pmatrix} \phi^{(1)} \\ \mathbf{0} \end{pmatrix} = \frac{1}{\sqrt{N_u^{(1)}}} \begin{pmatrix} \gamma + \delta \\ \alpha + i\beta \\ -\gamma^* + \delta^* \\ -\alpha^* - i\beta^* \end{pmatrix}, \quad (\text{A.7a})$$

$$u^{(2)}(p) = \frac{1}{\sqrt{N_u^{(2)}}} U^{(1)\dagger}(\tilde{E}_\psi^{(>)}, m_\psi, \mathbf{p}) \begin{pmatrix} \phi^{(2)} \\ \mathbf{0} \end{pmatrix} = \frac{1}{\sqrt{N_u^{(2)}}} \begin{pmatrix} \alpha - i\beta \\ -\gamma + \delta \\ -\alpha^* + i\beta^* \\ \gamma^* + \delta^* \end{pmatrix}, \quad (\text{A.7b})$$

$$N_u^{(1)} = N_u^{(2)} = \frac{1}{m_\psi} [(\text{Re } n^0)^2 - (\text{Im } n^0)^2 - (\text{Re } \mathbf{n})^2 + (\text{Im } \mathbf{n})^2], \quad (\text{A.7c})$$

and their completeness relation is given by:

$$\sum_{\alpha=1,2} u^{(\alpha)}(p) \bar{u}^{(\alpha)}(p) = \frac{1}{N_u^{(1)}} \begin{pmatrix} M_0 & 0 & M_+ & -M^* \\ 0 & M_0 & -M & M_- \\ M_- & M^* & M_0^* & 0 \\ M & M_+ & 0 & M_0^* \end{pmatrix}, \quad (\text{A.8a})$$

$$M_0 = n^2, \quad M_0^* = (n^*)^2, \quad (\text{A.8b})$$

$$M_+ = |\alpha|^2 + |\beta|^2 - 2\text{Im}(\alpha\beta^*) + |\gamma|^2 + |\delta|^2 + 2\text{Re}(\gamma\delta^*), \quad (\text{A.8c})$$

$$M_- = |\alpha|^2 + |\beta|^2 + 2\text{Im}(\alpha\beta^*) + |\gamma|^2 + |\delta|^2 - 2\text{Re}(\gamma\delta^*), \quad (\text{A.8d})$$

$$M = (\alpha^* + i\beta^*)(\gamma - \delta) - (\alpha + i\beta)(\gamma^* + \delta^*), \quad (\text{A.8e})$$

$$M^* = (\alpha - i\beta)(\gamma^* - \delta^*) - (\alpha^* - i\beta^*)(\gamma + \delta). \quad (\text{A.8f})$$

With the coefficients of Eq. (A.6b) one can show that

$$\begin{aligned} \sum_{\alpha=1,2} u^{(\alpha)}(p) \bar{u}^{(\alpha)}(p) &= \begin{pmatrix} m_\psi - i\hat{f}^{(6)} & 0 & E_\psi - p_3 & -(p_1 - ip_2) \\ 0 & m_\psi - i\hat{f}^{(6)} & -(p_1 + ip_2) & E_\psi + p_3 \\ E_\psi + p_3 & p_1 - ip_2 & m_\psi + i\hat{f}^{(6)} & 0 \\ p_1 + ip_2 & E_\psi - p_3 & 0 & m_\psi + i\hat{f}^{(6)} \end{pmatrix} + \mathcal{O}[(\hat{f}^{(6)})^2] \\ &= \not{p} + m_\psi \mathbf{1}_4 + i\hat{f}^{(6)} \gamma^5 + \mathcal{O}[(\hat{f}^{(6)})^2] = \not{p} + m_\psi \mathbf{1}_4 + \hat{\mathcal{Q}} + \mathcal{O}[(\hat{f}^{(6)})^2]. \end{aligned} \quad (\text{A.9})$$

Now the negative energy spinors are

$$v^{(1)}(p) = \frac{1}{\sqrt{N_v^{(1)}}} U^{(1)\dagger}(\tilde{E}_\psi^{(>)}, m_\psi, -\mathbf{p}) \begin{pmatrix} \mathbf{0} \\ \chi^{(1)} \end{pmatrix} = -\frac{1}{\sqrt{N_v^{(1)}}} \begin{pmatrix} \gamma + \delta \\ \alpha + i\beta \\ \gamma^* - \delta^* \\ \alpha^* + i\beta^* \end{pmatrix}, \quad (\text{A.10a})$$

$$v^{(2)}(p) = \frac{1}{\sqrt{N_v^{(2)}}} U^{(1)\dagger}(\tilde{E}_\psi^{(>)}, m_\psi, -\mathbf{p}) \begin{pmatrix} \mathbf{0} \\ \chi^{(2)} \end{pmatrix} = -\frac{1}{\sqrt{N_v^{(2)}}} \begin{pmatrix} \alpha - i\beta \\ -\gamma + \delta \\ \alpha^* - i\beta^* \\ -(\gamma^* + \delta^*) \end{pmatrix}, \quad (\text{A.10b})$$

$$N_v^{(1)} = N_v^{(2)} = \frac{1}{m_\psi} [(\text{Re } n^0)^2 - (\text{Im } n^0)^2 - (\text{Re } \mathbf{n})^2 + (\text{Im } \mathbf{n})^2], \quad (\text{A.10c})$$

and with the matrix elements of Eq. (A.8b) and the coefficients of Eq. (A.6b) one obtains:

$$\begin{aligned}
\sum_{\alpha=1,2} v^{(\alpha)}(p) \bar{v}^{(\alpha)}(p) &= \frac{1}{N_v^{(1)}} \begin{pmatrix} -M_0 & 0 & M_+ & -M^* \\ 0 & -M_0 & -M & M_- \\ M_- & M^* & -M_0^* & 0 \\ M & M_+ & 0 & -M_0^* \end{pmatrix} \\
&= \begin{pmatrix} -m_\psi + i\widehat{f}^{(6)} & 0 & E_\psi - p_3 & -(p_1 - ip_2) \\ 0 & -m_\psi + i\widehat{f}^{(6)} & -(p_1 + ip_2) & E_\psi + p_3 \\ E_\psi + p_3 & p_1 - ip_2 & -m_\psi - i\widehat{f}^{(6)} & 0 \\ p_1 + ip_2 & E_\psi - p_3 & 0 & -m_\psi - i\widehat{f}^{(6)} \end{pmatrix} + \mathcal{O}[(\widehat{f}^{(6)})^2] \\
&= \not{p} - m_\psi \mathbb{1}_4 - i\widehat{f}^{(6)} \gamma^5 + \mathcal{O}[(\widehat{f}^{(6)})^2] = \not{p} - m_\psi \mathbb{1}_4 - \widehat{\mathcal{Q}} + \mathcal{O}[(\widehat{f}^{(6)})^2].
\end{aligned} \tag{A.11a}$$

This completes the derivation of the results given by Eq. (6.44).

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